Scale-free dynamics in Internet traffic - The benefits of multivariate analysis ?

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Outline

Scale free Internet Traffic

Multivariate SelfSimilarity

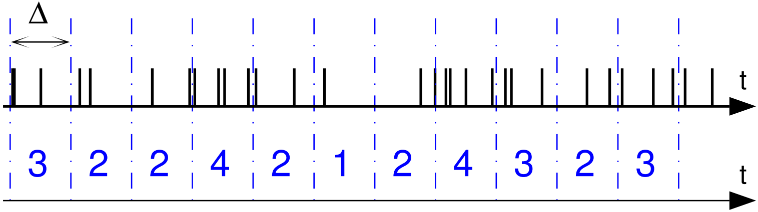
Multivariate Traffic

Anomaly detection

Conclusions

Aggregated Time Series

- Aggregation procedure
  - aggregation scale Δ,
  - Pkt or Byte counts.



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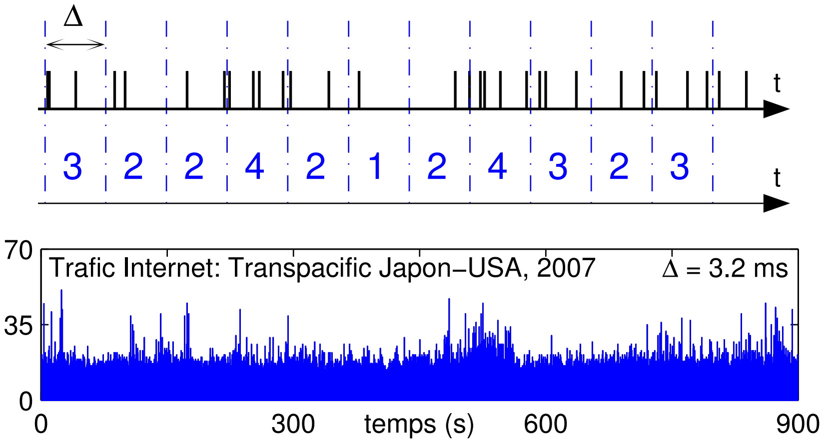
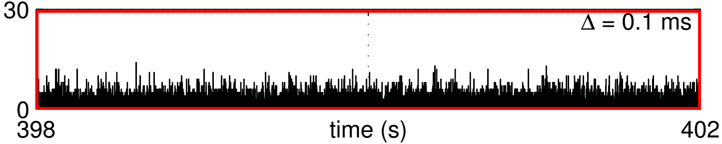


Scaling ? Covariance under aggregation



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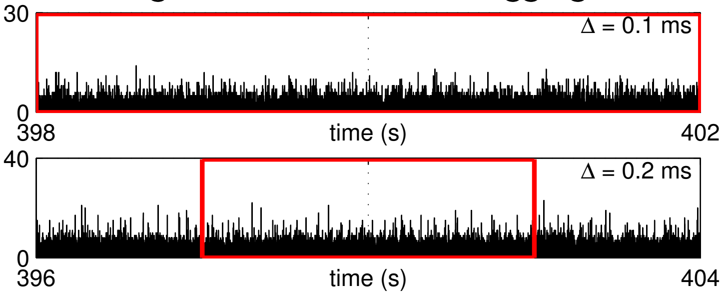

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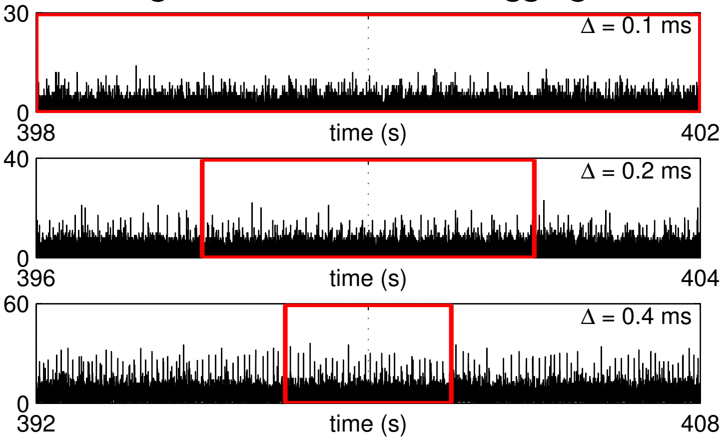

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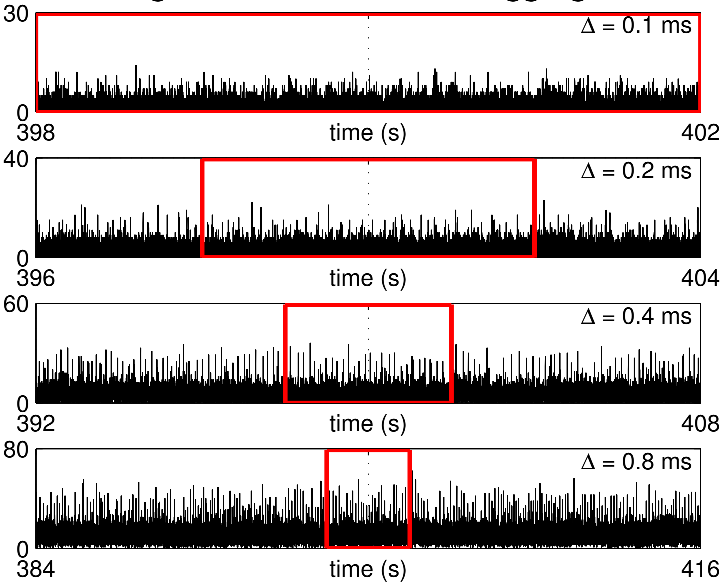
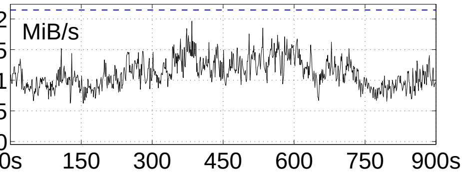
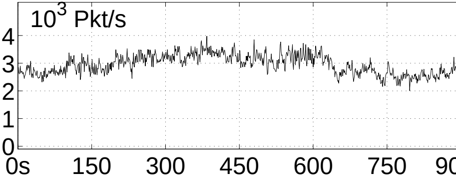


Statistical modeling of Internet traffic time series

- Aggregated time series: (aggregation levels Δ)
  - Packet counts, Byte counts, Flow counts,
  - Arrivals, durations, ...





- Statistics: $\Rightarrow$ *Irregularity, Burstiness !*
  - Long Range Dependence (covariance functions) ▸ Definition
  - Heavy Tails (Marginal Distributions) ▸ Definition

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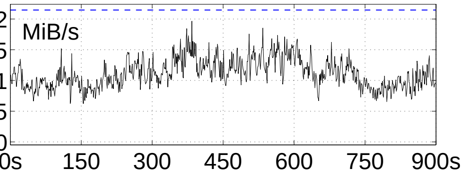

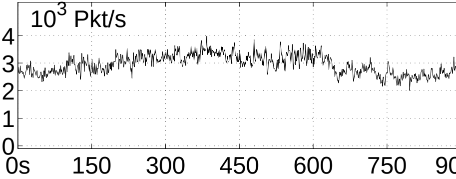

- Statistics: $\Rightarrow$ *Irregularity, Burstiness !*
  - Long Range Dependence (covariance functions) Definition
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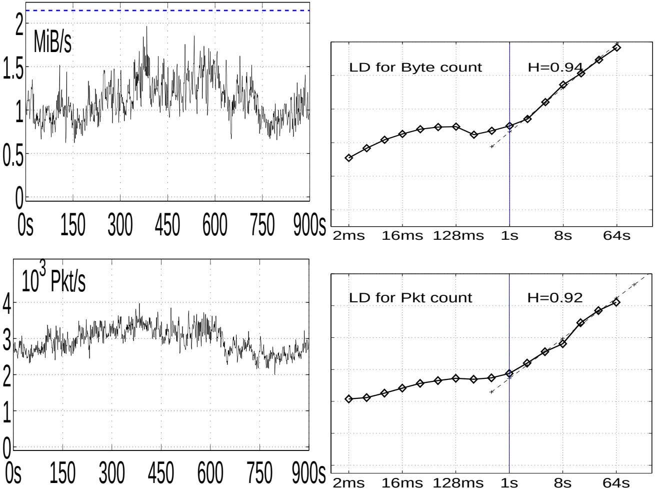
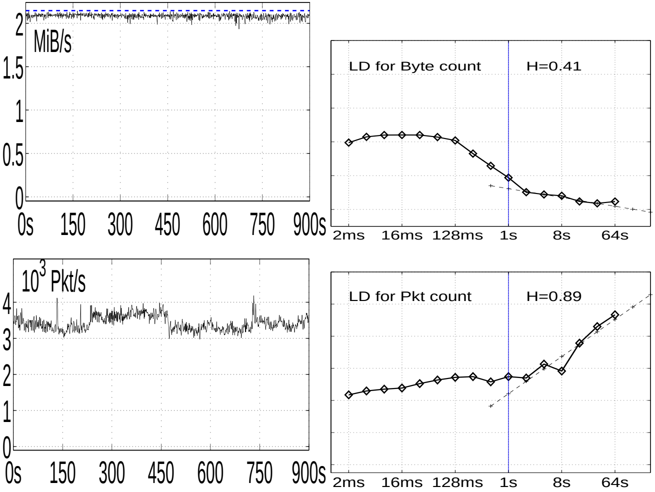
MAWI data: **B**-US2Jp, 2005/07/11



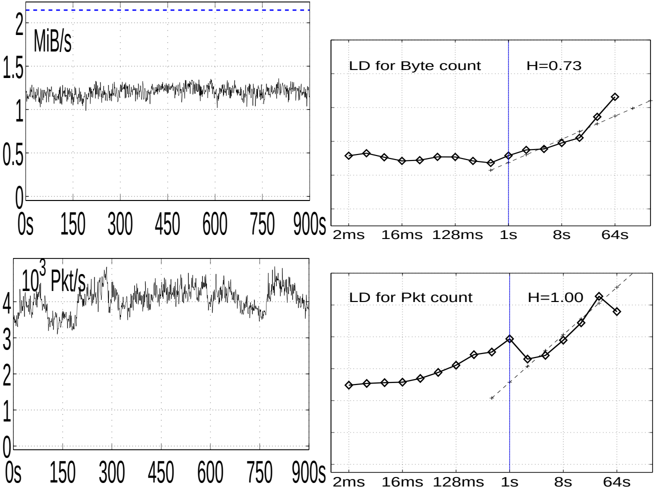
- Compares well with current knowledge and *theory/models*

MAWI data: **B**-US2Jp, 2003/06/03



- Congestion.

MAWI data: **B**-Jp2US, 2004/09/21



- Anomalies:

  network scan, spoofed flooding, attack on a Realserver

Random Projections or sketches

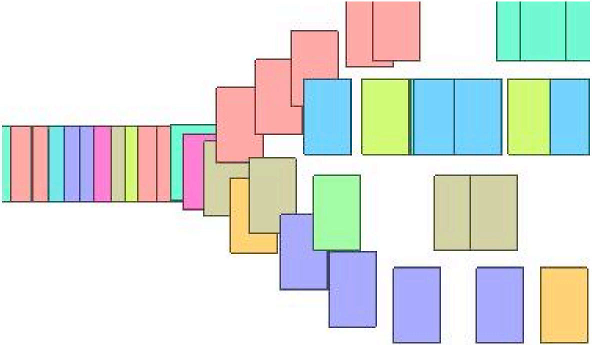
Sketches = ensemble of outputs of random hash table

[Muthukrishnan'03, Krishnamurty'03,...] [Abry+ SAINT'07, Dewaele+ Sigcomm LSAD'07]

- Random Hash Functions : h_n
  - $y = h(x)$,
  - $M-$ outputs: $y \in [1, \ldots, M]$,
  - $k-$ universal Hash functions.

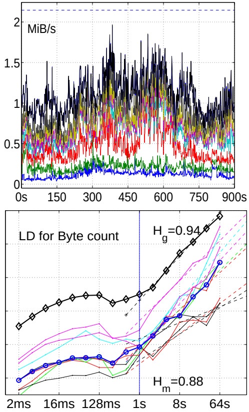
- Hash the Traffic :
  - Packet: $i-$th packet, $n-$tuple: $t_i, PTscr_i, PTdst_i, IPsrc_i, IPdst_i$
  - Choose one specific key: e.g., Destination Address
  - Hash according to this key: $m_i = h(IPdst_i) \in [1, \ldots, M]$,
  - All packets with same m_i = one sub-trace, sampled by random projection.
  - Aggregate traffic $\{t_i, m_i\}_{i \in I}$ into M series $X_\Delta^m(t)$, bins of Δs.

Sketched Traffic



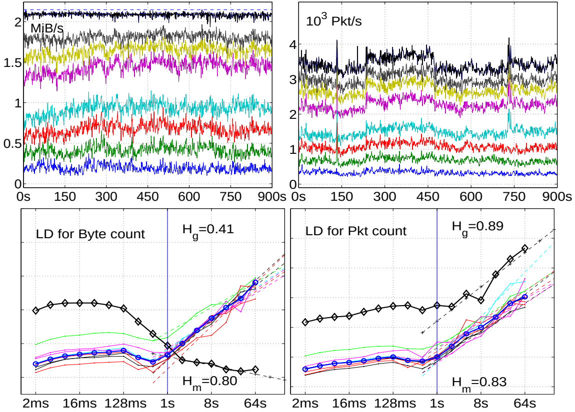
- Sketches = M sub-traces representing the total traffic
- Total of outputs = total trace (constrained sampling)
- Each sketched output = **random flow-sampling**

MAWI data: **B**-US2Jp, 2005/07/11



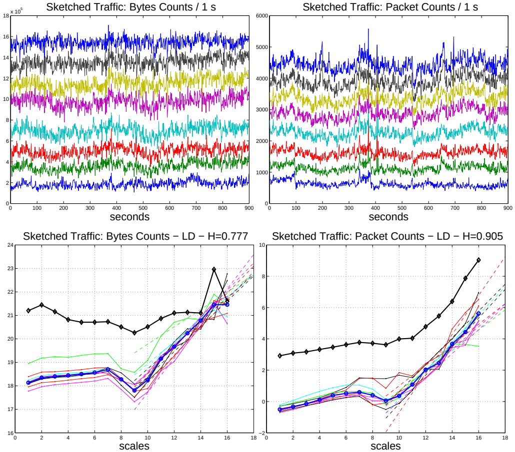
- All H_ms are consistent ! H_ms and H_g are consistent !
- LRDs on Bytes pr Pkts are consistent !
- Normal Traffic: no congestion (no anomaly ?)

MAWI data: **B**-US2Jp, 2003/06/03, Congestion



- $H_g^{\text{Byte}} \simeq 0.4$: no variability, no LRD, $H_g^{\text{Byte}} \neq H_g^{\text{Pkt}}$
- $H_m^{\text{Byte}} \simeq 0.9$, Flow variability, significant LRD, $H_m^{\text{Byte}} \simeq H_m^{\text{Pkt}}$

MAWI data: **B**-Jp2US, 2004/09/21, Anomalies

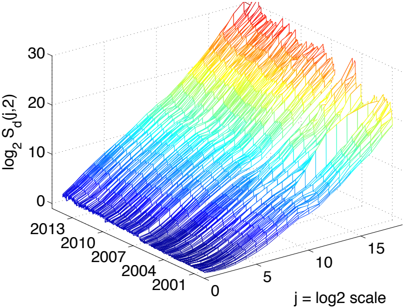


- $H_g^{\text{Byte}} \simeq 0.7$: LD ???, $H_g^{\text{Pkt}} \simeq 1$, ???
- $H_m^{\text{Byte}} \simeq 0.8$, LDs ok, significant LRD, $H_m^{\text{Byte}} \simeq H_m^{\text{Pkt}}$

Univariate Self-Similarity

Fontugne et al. 2017

- Long-Memory (Self-Similarity) at Coarse Scales, $H \simeq 0.9$.
- Multifractality like at Fine Scales
- Frontier scale around 1s, connected to RTT
- Random projections + Multiscale Analysis $\Rightarrow$ robust statistics, anomaly detection

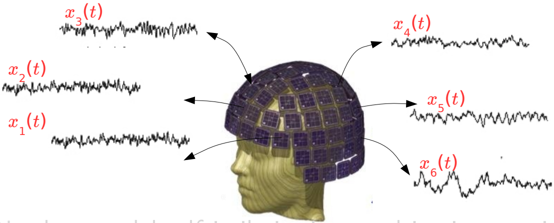


Another point of view?

Limitations

- Not versatile enough for data :
  - One-parameter model: $0 < H < 1$ - Jointly Gaussian
  - ⇒ Multifractal models (univariate) Mandelbrot 1974, Fontugne et al., 2017
  - ⇒ Non Gaussian asymptotically self-similar processes (univariate) Helgason et al., 2005
  - ⇒ Anisotropic SelfSimilar textures (univariate fields) Roux et al. 2013
- Data are naturally multivariate:

- Multivariate wavelet analysis: ‣ failure of univariate analysis



- Need to model selfsimilarity in a multivariate setting

Didier, Pipiras, 2011

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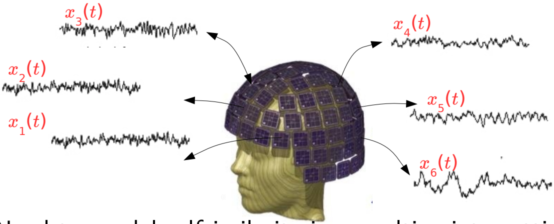
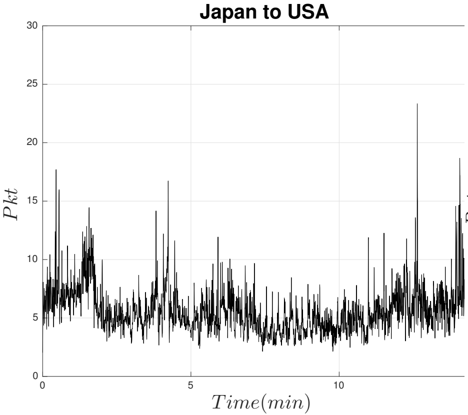


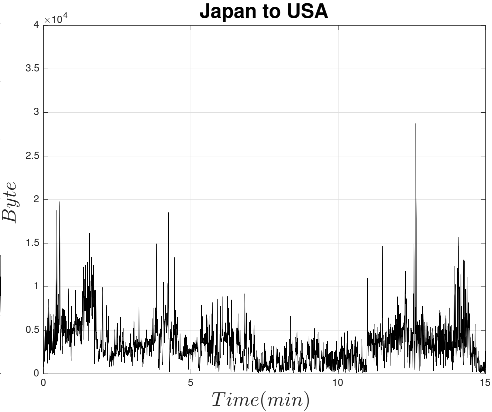
- Need to model selfsimilarity in a multivariate setting

Didier, Pipiras, 2011

Internet Traffic is naturally bivariate

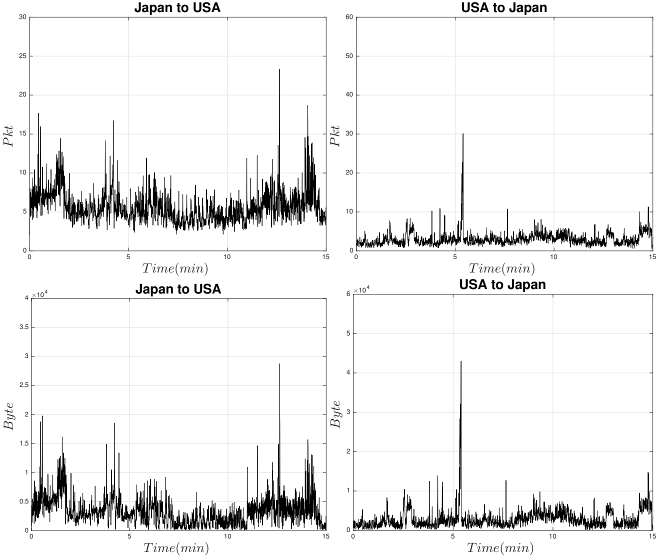
Fontugne et al. 2017, Abry, Didier 2017c





Internet Traffic is naturally 4-variate

Fontugne et al. 2017, Abry, Didier 2017c



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Multivariate SelfSimilarity

Multivariate Traffic

Anomaly detection

Conclusions

Operator Fractional Brownian Motion (OFBM): Definition

Didier, Pipiras, 2011

- M-components: $\{B_{\underline{H},\underline{\underline{\Sigma}}}(t)\}_{t \in \mathcal{R}}$

  $$\{B_{\underline{H},\underline{\underline{\Sigma}}}(t)\}_{t \in \mathcal{R}} = \{B_{h_1}(t), \ldots, B_{h_m}(t), \ldots B_{h_M}(t)\}_{t \in \mathcal{R}}$$

  - M-correlated fBm each with Hurst parameter $0 < h_m < 1$
    $\underline{H} = \{h_1, \ldots, h_m, \ldots, h_M\}$
  - $\underline{\underline{\Sigma}}$: $M \times M$ point covariance (positive definite) matrix
- Linear mixing:
  - $\underline{\underline{W}}$: $M \times M$ invertible matrix (in $\mathcal{R}^M$)
- OFBM: $t \in \mathcal{R} \to B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}} \in \mathcal{R}^M$
  - $B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}(t) = \underline{\underline{W}} \cdot B_{\underline{H},\underline{\underline{\Sigma}}}(t)$
  - *Free* parameters:

    $\underline{H}, \underline{\underline{\Sigma}}, \underline{\underline{W}}$

    $M + M(M-1)/2 + M(M-1) = 3/2M^2 + M/2$

Properties

- Covariance:

  - $\Sigma_{B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}}(t,t') \equiv W \Sigma_{B_{\underline{H},\underline{\underline{\Sigma}}}}(t,t') W^*$

$$(\Sigma_{B_{\underline{H},\underline{\underline{\Sigma}}}}(t,t'))_{m,m'} = (\underline{\underline{\Sigma}})_{m,m'} \cdot \left(|t|^{h_m+h_{m'}} + |t'|^{h_m+h_{m'}} - |t-t'|^{h_m+h_{m'}}\right)$$

$$\Rightarrow \underline{\underline{\Sigma}} \equiv \Sigma_{B_{\underline{H},\underline{\underline{\Sigma}}}}(1,1)$$

- Existence:

  - Matrix $G \circ \underline{\underline{\Sigma}}$ has full rank (Hadamard matrix product)
    $G_{m,m'} = \Gamma(h_m + h_{m'} + 1) \sin((h_m + h_{m'})\pi/2)$

  $\Rightarrow$ constraints on *Free* parameters:

  $\Rightarrow$ $\underline{H}$ and $\underline{\underline{\Sigma}}$ cannot be chosen independently

  $\Rightarrow$ e.g., $M = 2$: $\rho_{12} = \underline{\underline{\Sigma}}_{1,2} / sqrt(\underline{\underline{\Sigma}}_{1,1} \underline{\underline{\Sigma}}_{2,2})$

$$\Gamma(2h_1+1)\Gamma(2h_2+1)\sin(\pi h_1)\sin(\pi h_2) - \rho_{12}^2 \Gamma(h_1+h_2+1)^2 \sin^2(\pi(h_1+h_2)/2) > 0$$

- Time Reversibility:

  By definition: $\Sigma_{B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}}(t,t') = (\Sigma_{B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}}(-t,-t'))^T$

  There exist more general definitions

Multivariate SelfSimilarity

- Selfsimilarity:
  - $\{B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}(t)\}_{t\in\mathcal{R}} \stackrel{fdd}{=} \{a^{\underline{\underline{H}}} B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}(t/a)\}_{t\in\mathcal{R}}, \forall a > 0$
    where $\stackrel{fdd}{=}$: equality of all finite dimensional distributions,
    with $\underline{\underline{H}} = W \cdot$ Diag $\underline{H} \cdot W^{-1}$, $M \times M$ matrix
    where $a^{\underline{\underline{H}}} := \exp(\log(a\underline{\underline{H}})) = \sum_{k>0} \frac{(\log a\underline{\underline{H}})^k}{k\ !}$.

- Mixture of Power-laws:
  - when $W \equiv I_M$

$$\{B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}(t)\}_{t\in\mathcal{R}} \stackrel{fdd}{=} \{a^{h_1} B_{h_1}(t/a), \ldots, a^{h_m} B_{h_m}(t/a), \ldots a^{h_M} B_{h_M}(t/a)\}_{t\in\mathcal{R}}, \forall a > 0$$

  - when $W \neq I_M$
    Multivariate SelfSimilarity $\Rightarrow$ Mixtures of power-laws

Multivariate (discrete) Wavelet Transform

- Wavelet Coefficients:

$$D_{\mathrm{y_m}}(j,k) = \int_{\mathbb{R}} 2^{-j/2}\psi(2^{-j}t-k)Y_m(t)\mathrm{d}t$$

- Vector of Coefficients

$$D_{\mathrm{y}}(j,k) \equiv (D_{\mathrm{y_1}}(j,k),\ldots,D_{\mathrm{y_m}}(j,k),\ldots,D_{\mathrm{y_M}}(j,k))^T$$

- Wavelet Spectrum

$$S(2^j) = \frac{1}{K_j}\sum_{k=1}^{K_j} D(2^j,k)D(2^j,k)^*, \quad K_j = \frac{N}{2^j}$$

$S(2^j)$ is $M \times M$ matrix for each scale 2^j

N: data sample size

$$\mathbf{S(2^j)} = \begin{pmatrix} S_{\mathbf{11}}(2^j) & S_{\mathbf{12}}(2^j) & \ldots & \ldots & \ldots & S_{\mathbf{1}M}(2^j) \\ S_{\mathbf{21}}(2^j) & S_{\mathbf{22}}(2^j) & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & S_{\mathbf{33}}(2^j) & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{M\mathbf{1}}(2^j) & \ldots & \ldots & \ldots & \ldots & S_{MM}(2^j) \end{pmatrix},$$

Multivariate (discrete) Wavelet Transform and OFBM

Frecon et al. 2015, A., Didier 2017a, A., Didier 2017b, A., Didier 2017c,

- Short cuts:
  - Pre-Mixing: $X = B_{\underline{H},\underline{\underline{\Sigma}}}(t)\}_{t\in\mathcal{R}}$
  - Post-Mixing: $Y = B_{\underline{H},\underline{\underline{\Sigma}},\underline{\underline{W}}}(t)\}_{t\in\mathcal{R}}$
- Wavelet Coefficients

$$D_{\mathrm{y}}(j,k) = W2^{j(\underline{H}+I_M/2)}D_{\mathrm{x}}(0,k)$$

- Theoretical Wavelet Spectrum

$$\mathbb{E}D_{\mathrm{y}}(j,k)D_{\mathrm{y}}(j,k)^* = W2^{j(\underline{H}+I_M/2)}\mathbb{E}D_{\mathrm{x}}(0,k)D_{\mathrm{x}}(0,k)^*2^{j(\underline{H}+I_M/2)^*}W^*$$

$$(1)\ \mathbb{E}D_{\mathrm{y_m}}(j,k)D_{\mathrm{y_{m'}}}(j,k)^* = \sum_{p=1}^{M}\sum_{p'=1}^{M}A_{p,p'}^{(m,m')}(\underline{\underline{\Sigma}},\underline{\underline{W}})2^{j(h_p+h_{p'}+1)}$$

$\Rightarrow$ Mixtures of Power Laws

$\Rightarrow$ Identification: Non linear regression

Frecon et al. 2016, $M=2$, Branch and Bound Strategy

Multivariate analysis: Eigen Value Decomposition

Abry, Didier 2017a $M = 2$, Abry, Didier, Hui 2017b $\Sigma \equiv I_M$, Abry, Didier 2017c, $M \geq 2$

- For each scale j, :
  - Eigen Value Decomposition of $S(2^j)$:
    $$S(2^j) = U(2^j)\,\Lambda(2^j)\,U^*(2^j)$$

$$\mathbf{S}(\mathbf{2^j}) = \mathbf{U}(\mathbf{2^j}) \begin{pmatrix} \lambda_{\mathbf{1}}(S(2^j)) & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_{\mathbf{2}}(S(2^j)) & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_{\mathbf{3}}(S(2^j)) & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_M(S(2^j)) \end{pmatrix} \mathbf{U}(\mathbf{2^j})^{\mathbf{T}}$$

Multivariate analysis of SelfSimilarity

Abry, Didier 2017a $M = 2$, Abry, Didier 2017c, $M \geq 2$

- Assume:
  - $\forall (m, m'), m' \neq m, h_m \neq h_{m'}$
  - $0 < h_1 < \ldots < h_m < \ldots h_M < 1$

- Consistency:
  - $\lambda_m(S(2^{j(n)})) \to_{n \to +\infty} \xi_m 2^{2h_m j(n)}, \forall m = 1, \ldots, M$
  - $u_m \in \text{span}\{W_{\cdot,m}, W_{\cdot,m+1}, \ldots, W_{\cdot,M}\}, \quad 1 \leq m \leq M$

- Asymptotic Normality:

$$\sqrt{\frac{n}{2^{j(n)}}}\{\log_2 \lambda_m(S(2^{j(n)})) - \log_2 \lambda_m(\mathbb{E}S(2^{j(n)}))\}_{(m=1,\ldots,M, j_1(n) \leq j \leq j_2(n))} \to_{n \to +\infty} \mathcal{N}(0, \Sigma_\lambda)$$

Multivariate EVD estimation of Hurst exponents

- Multivariate estimators:

  $$\widehat{h}_m = \frac{1}{2}\sum_{j=j_1}^{j_2} w_j \log_2 \lambda_m(S(2^j))$$

- Asymptotic Normality:

  $$\sqrt{\frac{n}{2^j(n)}}\{\widehat{h}_m - h_m\}_{m=1,\dots,M} \rightarrow_{n\rightarrow+\infty} \mathcal{N}(0, M_{j_1,j_2}\Sigma_\lambda M^*_{j_1,j_2})$$

- Scaling range $(j_1(n), j_2(n)$

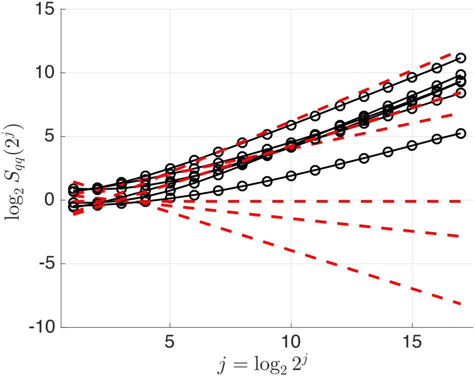
  $$(j_1(n), j_2(n) = (j_1^0 + f(n), (j_2^0 + f(n))$$ (see later)

- Univariate estimators:

  $$\widehat{h}_m^U = \frac{1}{2}(\sum_{j=j_1}^{j_2} w_j \log_2 S_{mm}(2^j) - 1)$$

Univariate (discrete) Wavelet Transform and OFBM

- Diagonal entries of $S_{m,m}(2^j)$:

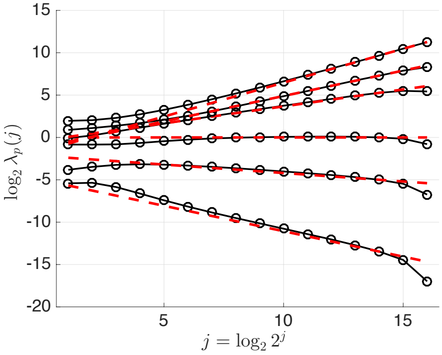


  - Mixture of Power-Laws
  - Dominant *h* only

$\Rightarrow$ Misleading conclusion: All *h* are equal

Multivariate (EVD) Wavelet Transform and OFBM

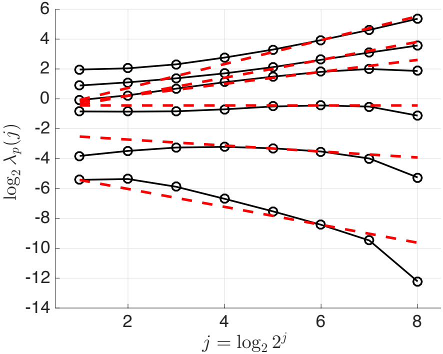
- Eigen Values of $S(2^j)$: λ_m



- Demixed Power-Laws
- All hs
- $\Rightarrow$ correct conclusion: All h can be different

Multivariate (EVD) Wavelet Transform and OFBM

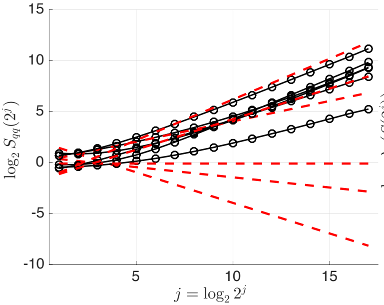
- Eigen Values of $S(2^j)$: λ_m



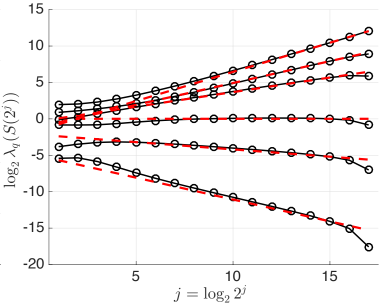
- Demixed Power-Laws
- All *h*s
- Even for very small sample size !

Multivariate (EVD) Wavelet Transform and OFBM

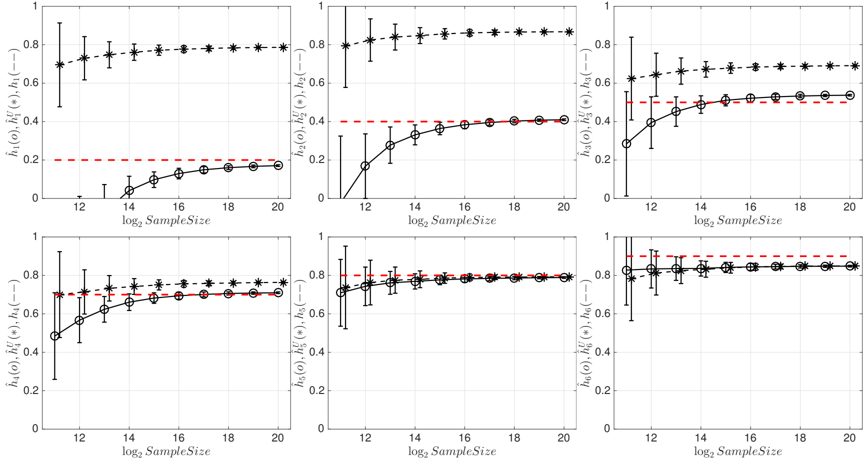
Diagonal entries of $S_{m,m}(2^j)$



Eigen Values of $S(2^j)$: λ_m



Estimation Performance: Bias $\rightarrow$ 0



▸ Proof

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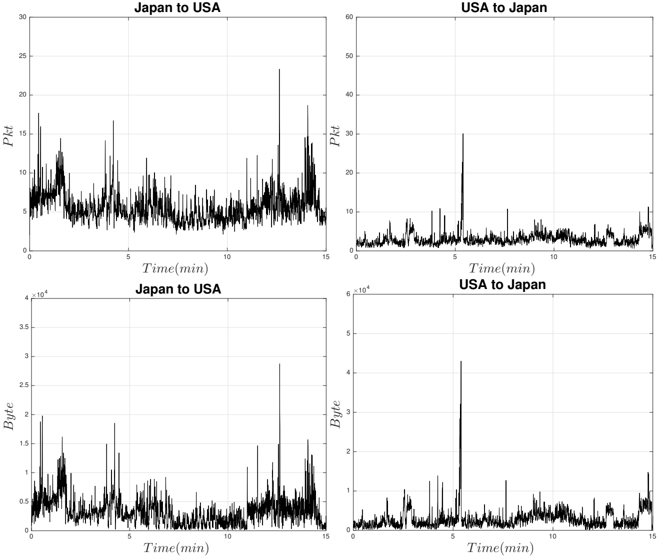
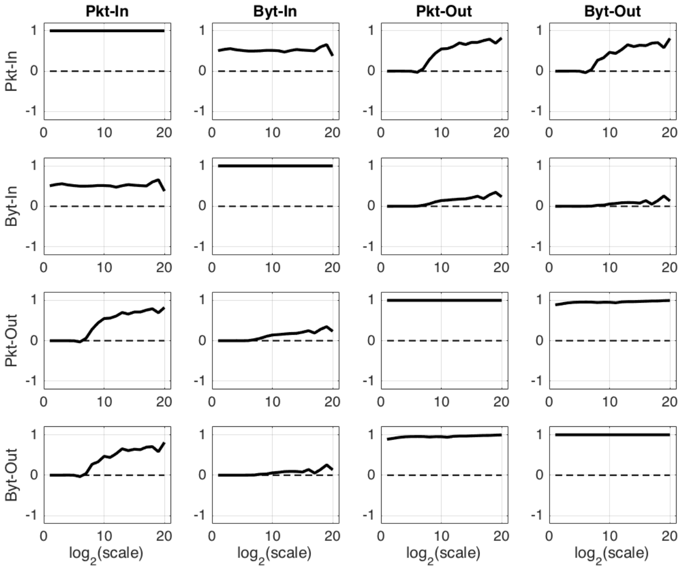
Internet Traffic - $M = 4$

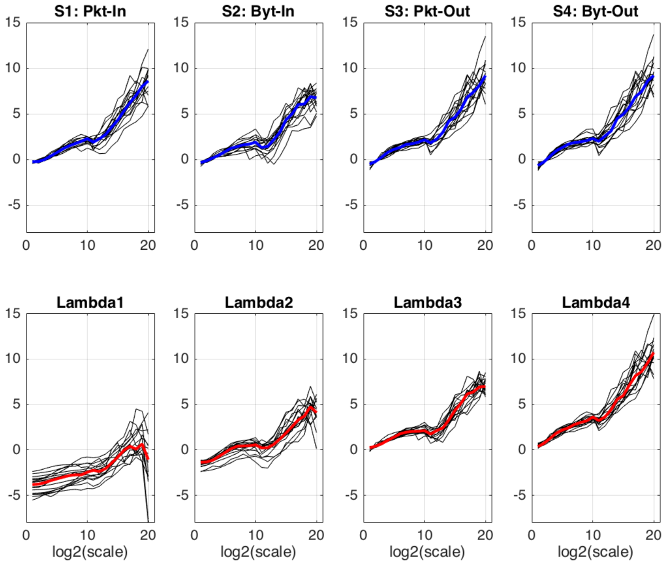
Fontugne et al. 2017, Abry, Didier 2017c



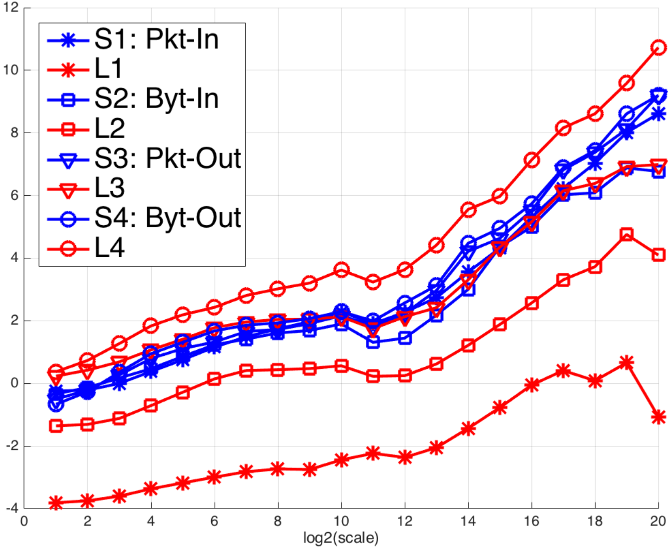
Wavelet Cross Coherence



Wavelet Eigen Structure and Random Projections



Multivariate (WavEigen) vs. Univariate Structures



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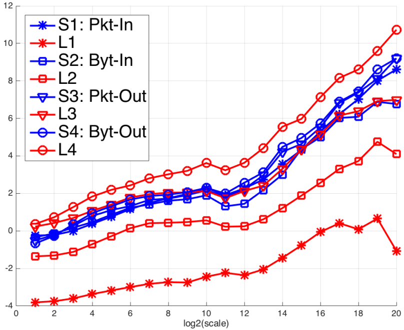
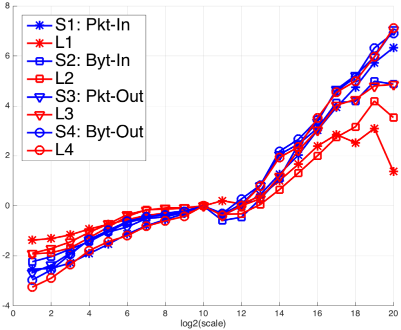
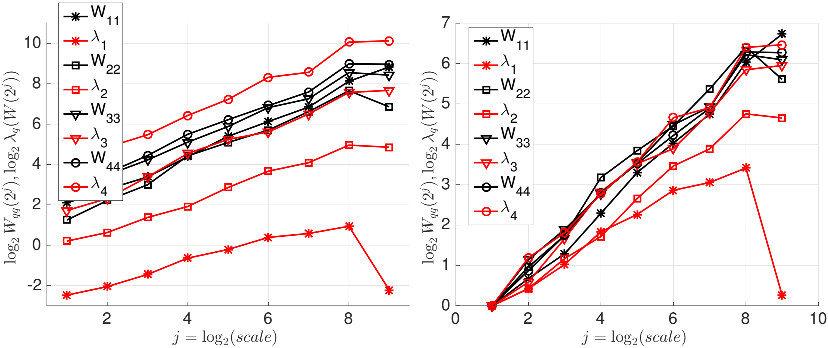


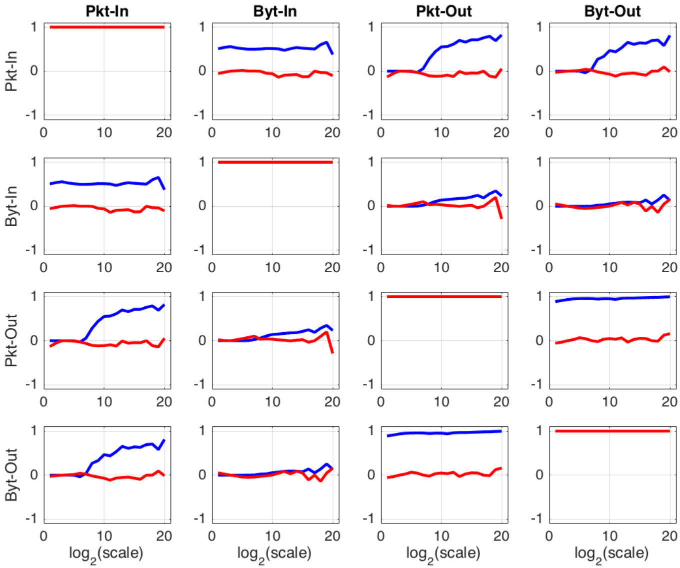


Long Memory at Coarse Scales

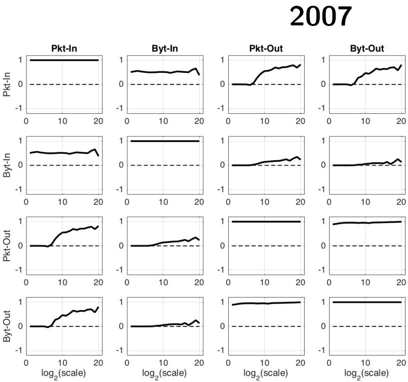


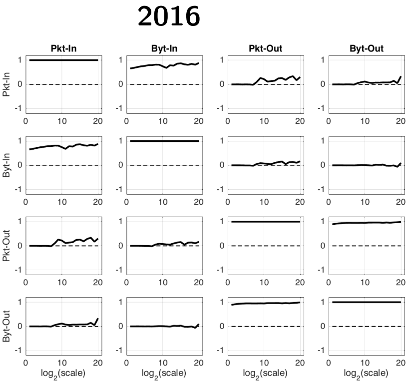
	$\widehat{H}_1$	$\widehat{H}_2$	$\widehat{H}_3$	$\widehat{H}_4$
univariate	0.85	0.86	0.86	0.90
multivariate	0.51	0.69	0.82	0.86

Demixing


Pkt-In
Byt-In
Pkt-Out
Byt-Out
Pkt-In
Byt-In
Pkt-Out
Byt-Out
log2(scale)

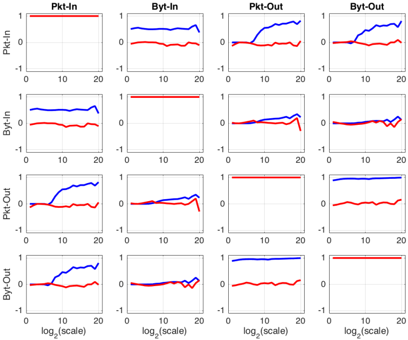
Wavelet Cross Coherence: 2007 vs 2016



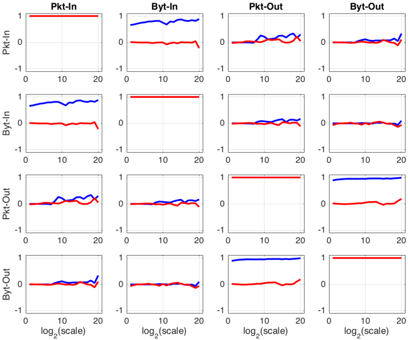


Demixing: 2007 vs 2016

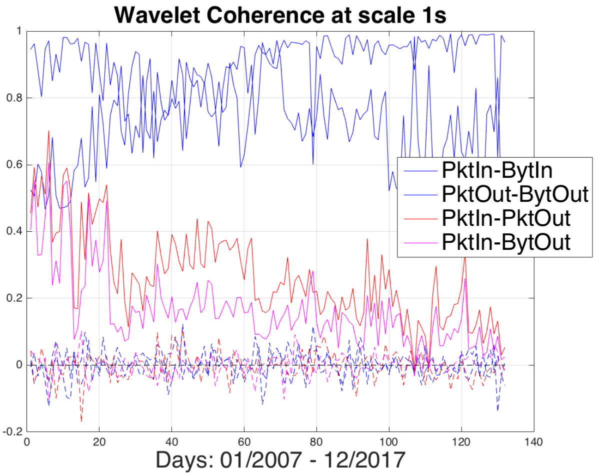
2007



2016

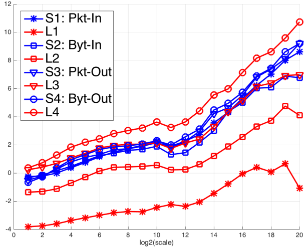


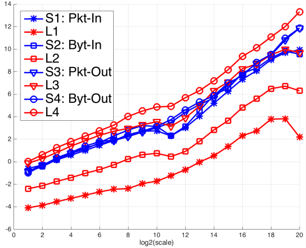
Wavelet Cross Coherence: from 2007 to 2017

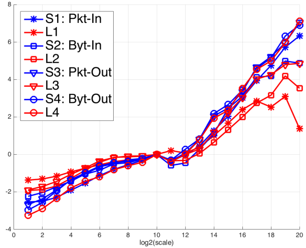


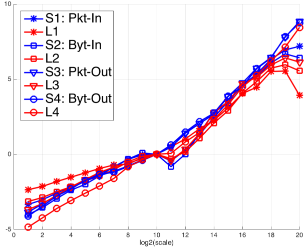
Multi vs. Uni Variate Structures: 2007 vs 2016

2007 | **2016**









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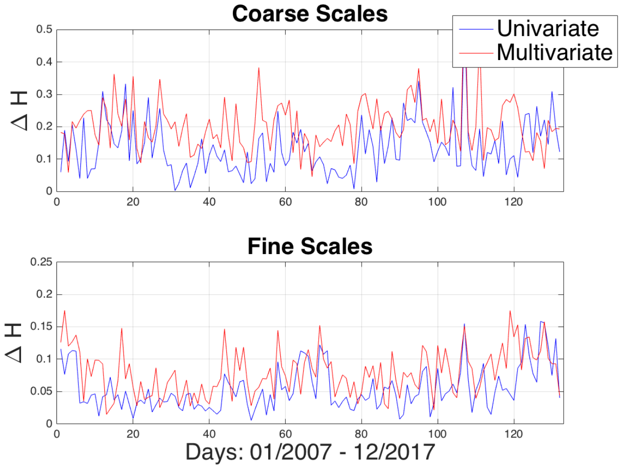


Outline

Scale free Internet Traffic

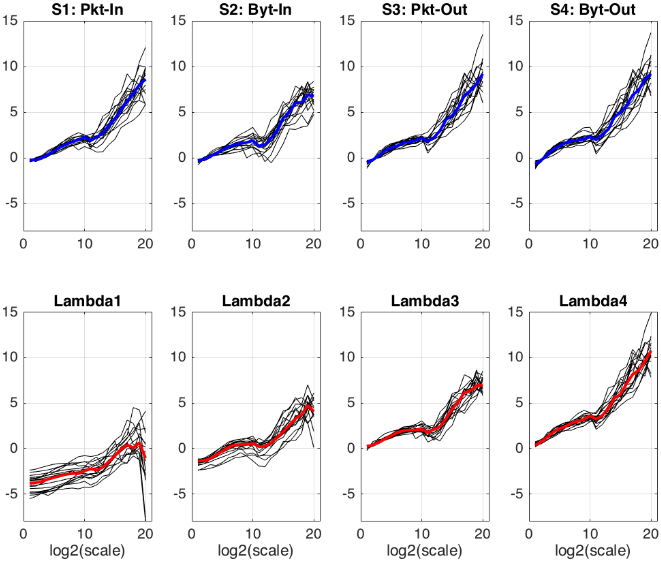
Multivariate SelfSimilarity

Multivariate Traffic

Anomaly detection

Conclusions

Principle


S1: Pkt-In
S2: Byt-In
S3: Pkt-Out
S4: Byt-Out
Lambda1
Lambda2
Lambda3
Lambda4
log2(scale)
15
10
5
0
-5
0
10
20

Case Study 1: Scan found by L1 only - Low Pkt

```
================================================
2007/02/15
scan found only by L1: src ip = "XXX.XXX.XXX.XXX"

1171515602.78838 (2007-02-15 14:00:02.078838)
1171516495.878712 (2007-02-15 14:14:55.878712)
nb_packets: 172
nb_bytes: 10664
src_addr: Nb different values: 1 (0 + 1)
Values:  "XXX.XXX.XXX.XXX" 172
dst_addr: Nb different values: 167 (162 + 5)
transport_portocol: Nb different values: 1 (0 + 1)
Values: tcp 172

TCP:
TCP: nb packets: 172
Src port: Nb different values: 164 (157 + 7)
Dst port: Nb different values: 2 (0 + 2)
Values: 139 74;5900 98
nb urg packets: 0
nb ack packet: 0
nb psh packet: 0
```

Case Study 2: Scan found by L2 only - Short Duration

```
=================================================
2007/01/15
scan found only by L2: src ip XXX.XXX.XXX.XXX

1168837324.539858 (2007-01-15 14:02:04.539858)
1168837473.391106 (2007-01-15 14:04:33.391106)
nb_packets: 1071
nb_bytes: 70686
src_addr: Nb different values: 1 (0 + 1)
Values: XXX.XXX.XXX.XXX 1071
dst_addr: Nb different values: 254 (0 + 254)
transport_portocol: Nb different values: 1 (0 + 1)
Values: tcp 1071

TCP:
TCP: nb packets: 1071
Src port: Nb different values: 480 (77 + 403)
Dst port: Nb different values: 2 (0 + 2)
Values: 139 549;445 522
nb urg packets: 0
nb ack packet: 0
nb psh packet: 0
nb rst packet: 0
```

Longitudinal Study

- Method:

  One day per month, from 2007 to 2017

  Trinocular: filtered out

- Detection:

  Top-5 most anomalous sketch,

  8 successive hash tables,

  Anomalous if suspicious in each hast table,

  Similarity index: $(|A \cap B|)/\min(|A|,|B|)$

Longitudinal Study - from 2007 to 2017

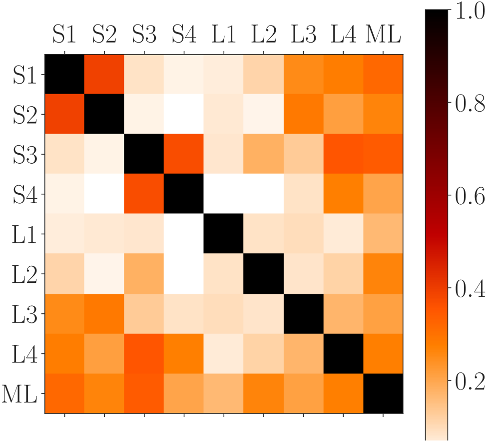
- MawiLab: ~ 142 detections per day on average
- Multiscale:

	1	2	3	4
S	~ 9	$\sim$8	~ 9	~ 8
L	~ 7	~ 7	~ 7	$\sim$9

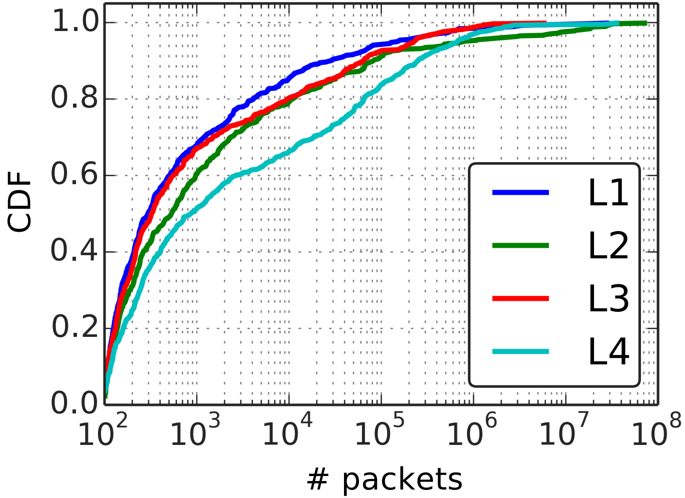
- Multiscale S $\cap$ L: 30% only !

Longitudinal Study

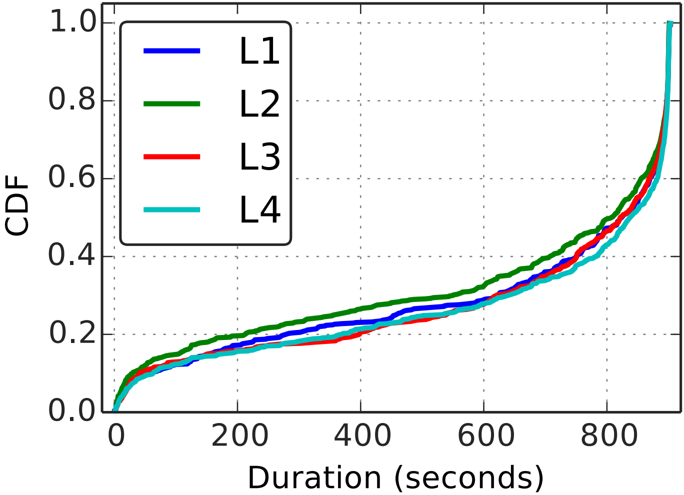
- Univariate: Same anomalies for Pkt & Byt in same direction
- Multivariate: L4 $\sim$ univariate, L1, L2, L3: different anomalies
- MawiLab: Univ. Pkt, then Univ. Byt
  much less in common with L1, L2, L3



Longitudinal Study - Low Pkt Anomalies



Longitudinal Study - Short Duration Anomalies



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Conclusions and perspectives

- Scale-free dynamics:
  - Ubiquitous in applications
  - Well-modeled by SelfSimilarity
  - Efficiently analyzed with wavelets
- Multivariate SelfSimilarity:
  - But Data are multivariate
  - Multivariate SelfSimilarity model (OFBM)
  - Multivariate wavelet analysis: ▸ Change of Perspectives

    Univariate: Scales then Components

    Multivariate: Components then Scales

    ⇒ Efficient and robust estimation procedures
- Internet data:
  - Longitudinal study ? Demixing ? Interpretation ?
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- References:
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References - Theory

- Multivariate SelfSimilarity Analysis:

- Frecon 2016: J. Frecon, G. Didier, N. Pustelnik, P. Abry, Non-Linear Wavelet Regression and Branch and Bound Optimization for the Full Identification of Bivariate Operator Fractional Brownian Motion, IEEE Transactions on Signal Processing, 64(15):4040-4049, 2016. .pdf
- Abry, Didier, 2017a: Abry, P. and Didier, G., Wavelet estimation for operator fractional Brownian motion, Bernoulli, to appear, 2017. .pdf
- Abry, Didier, Hui, 2017b: Abry, P., Didier, G., Hui L., Two-step wavelet-based estimation for mixed Gaussian fractional processes, Preprint, 2017. .pdf
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- H. Wendt, G. Didier, S. Combrexelle, P. Abry, Multivariate Hadamard self-similarity: testing fractal connectivity, Signal Processing, 2017. .pdf
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References - Applications

- Internet Traffic:

- Fontugne et al. 2017: R. Fontugne, P. Abry, K. Fukuda, D. Veitch, K. Cho, P. Borgnat, H. Wendt, Scaling in Internet Traffic: a 14 year and 3 day longitudinal study, with multiscale analyses and random projections, IEEE Trans. on Networking, 2017 .pdf
- P. Abry, R. Baraniuk, P. Flandrin, R. Riedi, D. Veitch, Multiscale Network Traffic Analysis, Modeling, and Inference Using Wavelets, Multifractals, and Cascades, IEEE Signal Processing Magazine 19(3):28–46, May 2002. .pdf

- Neurosciences:

- Ph. Ciuciu, P. Abry, B. He., Interplay between functional connectivity and scale-free dynamics in intrinsic fMRI networks, NeuroImage, 95:248-263, 2014. .www .pdf

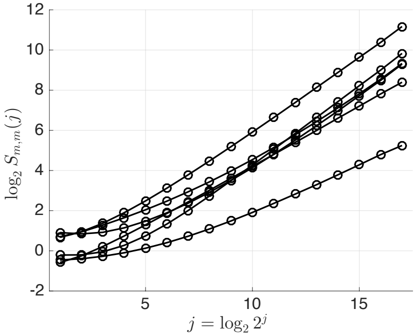
- Art Investigations:

- P. Abry, S. G. Roux, H. Wendt, P. Messier, A. G. Klein, N. Tremblay, P. Borgnat, S. Jaffard, B. Vedel, J. Coddington, L. Daffner, Multiscale Anisotropic Texture Analysis and Classification of Photographic Prints: Art scholarship meets image processing algorithms, IEEE Signal Processing Mag., vol. 32, no. 4, pp. 18-27, July 2015. .pdf

Thank you !

Univariate analysis is dangerous !

Univariate versus Multivariate Analyses

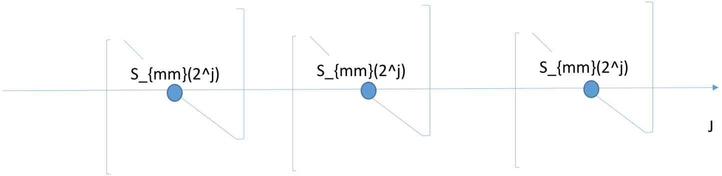


- Diagonal entries of $S_{m,m}(2^j)$:
  - Mixture of Power-Laws

$\Rightarrow$ Misleading conclusion: All h are equal

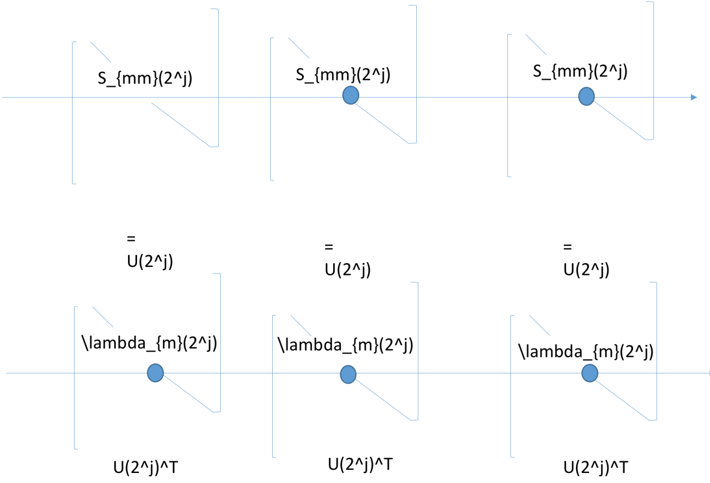
back

Univariate versus Multivariate Analyses



For each m
Study S_mm(2^j)
As a function of j
Then scan m

Univariate versus Multivariate Analyses



For each component m
Study S_mm(2^j)
As a function of j
Then scann
across components

J

For each scale 2^j
Diagonalize S(2^j)
Then study
\lambda_{m}(2^j)
As a function of j

j

◂ back

Long Range Dependence (or covariance)

- Theory: Y 2nd order stationary process
  - Definition:

    Spectrum: $\Gamma_Y(\nu) \sim C_\Gamma |\nu|^{-\gamma}$, $|\nu| \to 0$, $0 < \gamma < 1$,
    Covariance: $\gamma_Y(\tau) \sim C_\gamma |\tau|^{-(1-\gamma)}$, $|\tau| \to \infty$
  - Self-similarity:

    X is H-ss, $\{X(t), t \in \mathcal{R}\} \stackrel{fdd}{=} \{a^H X(t/a), t \in \mathcal{R}\}$, $a > 0$,
    if stationary increments, $Y(k) = X(k+1) - X(k)$
    and $1/2 < H < 1$,
    then Y is LRD with $\gamma = 2H - 1$.
  - Fractional Brownian motion $B_H(t)$:
    Gaussian H-ss, with stationary increments

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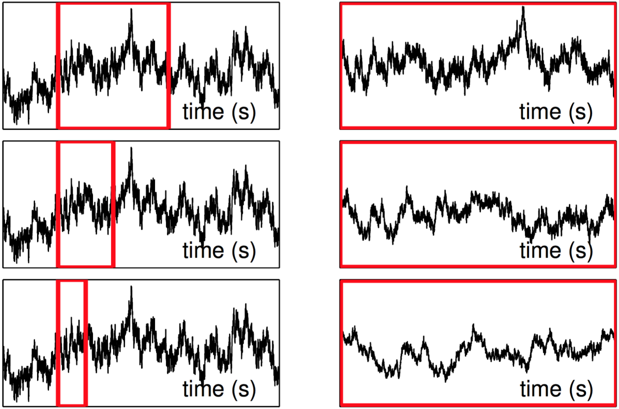
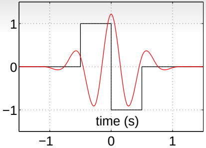
Scale-free dynamics: Intuition



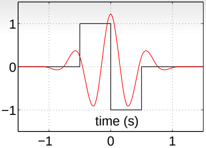
- Covariance under Dilation (Change of Scale),
- The Whole and the SubPart (Statistically) Undistinguishable,
- No Characteristic Scale of Time

LRD and Wavelets



- Wavelets: ▸ Wavelet Transform
  - Mother-Wavelet ψ: Oscillating pattern,
  - Number of vanishing moments N_ψ: $\forall k = 0, ..., N-1$, $\int_{\mathcal{R}} t^k \psi_0(t) dt \equiv 0$ and $\int_{\mathcal{R}} t^N \psi_0(t) dt \neq 0$.
  - Basis: $\{\psi_{j,k}(t) = 2^{-j/2}\psi_0(2^{-j}t - k)\}$,
  - Coefficients of Y: $d_Y(j,k) = \langle \psi_{j,k}, Y \rangle$
- Wavelets and 2nd order stationary process:
  - $\mathbf{E}|d_Y(j,k)|^2 = \int_{\mathcal{R}} \Gamma_Y(\nu) 2^j |\tilde{\Psi}_0(2^j \nu)|^2 d\nu$,
- Wavelets and LRD:
  - $\mathbf{E}|d_Y(j,k)|^2 \sim C 2^{j(2H-1)}$ for $2^j \to +\infty$,
  - $S(j) = \frac{1}{n_j} \sum_k |d_Y(j,k)|^2$,
  - Logscale Diagram: $\log_2 S(j)$ vs. $\log_2 2^j = j$,
  - $\hat{H} = \frac{1}{2}\left(1 + \sum_{j=j_1}^{j_2} w_j \log_2 S(j)\right)$.

LRD and Wavelets



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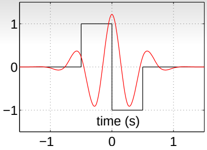

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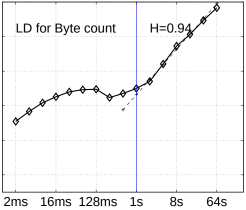

Wavelet Transform

- Let ψ_0 denote an elementary mother wavelet,
- Shifted and dilated templates of ψ_0:
  $\psi_{j,k}(t) = 2^{-j/2}\psi_0(2^{-j}t - k)$,
- Wavelet Coefficients: $d_{X_\Delta}(j,k) = \langle \psi_{j,k}, X_\Delta \rangle$.

◂ Back

Sketch of Proof - 1

- $\log_2 \lambda_m(S(2^j(n))) \to 2jh_m$?

- Courant-Fischer principle:
  - Let $\mathcal{U}_m$ such that $\dim \mathcal{U}_m = m$

$$\lambda_m(S(2^j)) = \inf_{\mathcal{U}_m} \sup_{x \in \mathcal{U}_m \cap S_{\mathbb{C}}^{M-1}} x^* S(2^j) x$$

- Hence:
  - Study $x^* S(2^j) x$

Sketch of Proof - 2

- Wavelet Spectrum:
  $\mathbb{E}D_{\mathrm{y}}(j,k)D_{\mathrm{y}}(j,k)^* = W2^{j(\underline{H}+I_M/2)}\mathbb{E}D_{\mathrm{x}}(0,k)D_{\mathrm{x}}(0,k)^*2^{j(\underline{H}+I_M/2)^*}W^*$

$$\mathbb{E}D_{\mathrm{y}}(j,k)D_{\mathrm{y}}(j,k)^* = W2^{j(\underline{H}+I_M/2)}\underbrace{W^{-1}\mathbb{E}D_{\mathrm{Y}}(0,k)D_{\mathrm{y}}(0,k)^*(W^*)^{-1}}_{B_{W,\Sigma}(0)}2^{j(\underline{H}+I_M/2)^*}W^*$$

$$S(2^j) = W2^{j(\underline{H}+I_M/2)}\underbrace{W^{-1}D_{\mathrm{Y}}(0,k)D_{\mathrm{y}}(0,k)^*(W^*)^{-1}}_{\hat{B}_{W,\Sigma}(0)}2^{j(\underline{H}+I_M/2)^*}W^*$$

- OFBM is well-defined:
  - $\Rightarrow$ $B_{W,\Sigma}(0)$ has bounded eigen values
  - $\Rightarrow$ $\hat{B}_{W,\Sigma}(0)$ has bounded eigen values
  - $\Rightarrow$ $0 < A \leq \lambda_m(\hat{B}_{W,\Sigma}(0)) \leq B < \infty$
- Hence:
  $0 < A \cdot x^*WDD^*W^*x \leq x^*S(2^j)x \leq B \cdot x^*WDD^*W^*x < \infty$

  with $D = \mathrm{Diag}\{2^{jh_1}, \ldots, 2^{jh_m}, \ldots, 2^{jh_M}\}$

Sketch of Proof - 4

- $D = \text{Diag}\{2^{jh_1}, \dots, 2^{jh_m}, \dots, 2^{jh_M}\}$
- When $W = I_M$:

$$0 < A \cdot x^* WDD^* W^* x \leq x^* S(2^j) x \leq B \cdot x^* WDD^* W^* x < \infty$$

$$0 < A \cdot x^* DD^* x \leq x^* S(2^j) x \leq B \cdot x^* DD^* x < \infty$$

$$\forall m = 1, \dots, M,\ 0 < A \cdot \lambda_m(DD^*) \leq \lambda(S(2^j)) \leq B \cdot \lambda_m(DD^*) < \infty$$

$$\forall m = 1, \dots, M,\ 0 < A \cdot 2^{2jh_m} \leq \lambda_m(S(2^j)) \leq B \cdot 2^{2jh_m} < \infty$$

$$\Rightarrow\ \forall m = 1, \dots, M,\ \log_2 \lambda_m(S(2^j))) \to 2h_m \cdot j$$

Sketch of Proof - 5

- When $W \neq I_M$:
  $$0 < A \cdot x^* W D D^* W^* x \leq x^* S(2^j) x \leq B \cdot x^* W D D^* W^* x < \infty$$

$$x^* W D D^* W^* x = \frac{x^* W}{||W^* x||} D D^* \frac{W^* x}{||W^* x||} \times ||W^* x||^2$$

$$0 < A' \cdot \leq ||W^* x||^2 \leq B' < +\infty \text{ since } W \text{ invertible}$$

$$0 < A' \cdot \frac{x^* W}{||W^* x||} D D^* \frac{W^* x}{||W^* x||} \leq x^* W D D^* W^* x < B' \cdot \frac{x^* W}{||W^* x||} D D^* \frac{W^* x}{||W^* x||} < +\infty$$

$$\forall m = 1, \ldots, M, \; 0 < A \cdot A' \cdot \lambda_m(D D^*) \leq \lambda(S(2^j)) \leq B \cdot B' \cdot \lambda_m(D D^*) < \infty$$

$$\forall m = 1, \ldots, M, \; 0 < A \cdot A' \cdot 2^{2jh_m} \leq \lambda_m(S(2^j)) \leq B \cdot B' \cdot 2^{2jh_m} < \infty$$

$\Rightarrow$ $\forall m = 1, \ldots, M, \; \log_2 \lambda_m(S(2^j))) \to 2h_m \cdot j$
$\Rightarrow$ $W \neq I_M$ does not create difficulties compared to $W \equiv I_M$
$\Rightarrow$ key step in proof: $B_{W,\Sigma}(0)$ has bounded eigen values

Sketch of Proof - 6

- $M = 2$
  $$\lambda_1(2^j) = 2\frac{\det(\mathbf{E}S)}{\mathbf{E}S_{22}(1)}\left(2^{2jh_1} + \frac{\mathbf{E}S_{22}(1)}{F(\mathbf{E}S_{11}(1),\mathbf{E}S_{22}(1))}2^{2j(h_1-h_2)}\right)$$

- Scaling range $(j_1(n), j_2(n)$
  $$(j_1(n), j_2(n) = (j_1^0 + f(n), (j_2^0 + f(n))$$
  $$\beta \log_2 n \leq f(n) \leq (1-\epsilon)\log_2 n,\ \epsilon > 0$$
  $$\beta = \frac{1}{1+2\max(h_1, \min_{1\leq m<m'\leq M}(h_{m'}-h_m))}$$
  Bias-Variance trade-off

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